

Extension of germs of holomorphic foliations

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Abstract

We consider the problem of extending germs of plane holomorphic foliations to foliations of compact surfaces. We show that the germs that become regular after a single blow up and admit meromorphic first integrals can be extended, after local changes of coordinates, to foliations of compact surfaces. We also show that the simplest elements in this class can be defined by polynomial equations. On the other hand we prove that, in the absence of meromorphic first integrals there are uncountably many elements without polynomial representations.¹

1 Introduction

In this paper we treat the following problem: let $\mathcal{Fol}(\mathbb{C}^2, 0)$ be the set of germs of holomorphic foliations defined in a neighborhood of $0 \in \mathbb{C}^2$ which are singular at the origin; we consider two such foliations to be equivalent when they are conjugated by a local holomorphic diffeomorphism of \mathbb{C}^2 at $0 \in \mathbb{C}^2$. We select some family $L \subset \mathcal{Fol}(\mathbb{C}^2, 0)$ and ask if an equivalence class contains a foliation that is defined by polynomial differential equations. Any member of such a class admits an extension to a foliation of the complex projective plane after a suitable local change of coordinates.

A very simple example is given by the set of hyperbolic singularities in the Poincaré domain, namely, foliations defined by the 1-form

$$(x + A(x, y)) dy - (\lambda y + B(x, y)) dx = 0,$$

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where $\lambda \notin \mathbb{R}$ and A and B are holomorphic functions such that $A(0,0) = B(0,0) = 0$ and whose derivatives at $0 \in \mathbb{C}$ vanish. By the theorem of linearisation of Poincaré ([9]), we have holomorphic equivalence to the linear part

$$x dy - \lambda y dx = 0.$$

It is interesting to notice that when $\lambda \leq 0$ but $\lambda \notin \mathbb{Q}$ it is not known if any equivalence class contains a foliation defined by polynomial equation (see [8]).

In their paper [5], Genzmer and Teyssier introduced a tool that allows to treat this problem from an analytic point of view. Roughly speaking the idea is: given the family L , a surjective map ψ from the set $[L]$ of equivalence classes in L to a space I of invariants is defined; it is assumed that there are appropriate topologies to turn ψ into an "analytic" map. The image of the equivalence classes of polynomially defined foliations is then a countable union of analytic manifolds (of finite dimension!) and cannot be the whole of I provided I is a "huge" space. This is a sort of analytic Baire property. As an application, the authors consider the family L of saddle-node singularities of Milnor number 2 (in fact the choice of the Milnor number is not relevant). Those are singularities defined by forms

$$[y(1 + \mu x) + R(x, y)] dx - x^2 dy = 0$$

where $\text{ord}_{(0,0)}(R) \geq 3$. According to [7], these singularities can be obtained applying a convenient gluing procedure to normal forms

$$y(1 + \mu x) dx - x^2 dy = 0;$$

in this gluing process non trivial local holomorphic diffeomorphisms $h(z) = z + \dots$ are used and become elements of the space of invariants I . The conclusion is that there are uncountably many saddle-nodes which are not equivalent to saddle-nodes defined by polynomial equations.

In the present paper we apply these ideas to the space T of singular foliations that become regular after a single blow-up. The lack of singularities implies that every leaf of the foliation intersects the exceptional divisor and a finite number of them are tangent to it. In order to simplify the exposition, we consider the case with just one point of tangency of order 1 with the exceptional divisor; let T_1 be the space of these singularities. We may then state

Theorem 1. There are foliations in T_1 which are not holomorphically equivalent to foliations defined by polynomial equations.

The space of invariants in this situation contains the local involutions at the tangency points which are naturally associated to the foliations in T_1 .

On the other hand, there are some situations where we may wonder if the foliation is conjugated to a polynomial model. For example, let us consider a germ of holomorphic function $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ such that $0 \in \mathbb{C}^2$ is an isolated singularity of f . By a theorem of Mather, f has finite determinacy: there exists $k \in \mathbb{N}$ such that the k -jet f_k of f at $0 \in \mathbb{C}^2$ is conjugated to f by means of a holomorphic diffeomorphism ϕ , i.e., $f = f_k \circ \phi$. It follows that the foliation defined by $df = 0$ is conjugated by ϕ to the foliation defined by $df_k = 0$. The question of whether a similar statement holds true for foliations defined by meromorphic functions arises: is a foliation in $(\mathbb{C}^2, 0)$ defined by meromorphic function equivalent to a polynomially defined foliation? A theorem by Cerveau and Mattei ([3]) gives sufficient conditions on the function to conclude that it is the case: let f/g be a germ at $0 \in \mathbb{C}^2$ of a meromorphic function (f and g are supposed to be relatively prime germs of holomorphic functions) such that the 1-form $f dg - g df$ defines a foliation with an isolated singularity at $0 \in \mathbb{C}^2$. Then f/g has finite determinacy, that is, f/g is conjugated to f_k/g_k for some $k \in \mathbb{N}$, where f_k and g_k are the k -jets of f and g at $0 \in \mathbb{C}^2$. The simplest examples of functions defining foliations in T_1 do not satisfy the hypothesis of Cerveau and Mattei's result - take for instance $\frac{y^2 - x^3}{x^2} = \text{const.}$ However we prove the following

Theorem 2. Any foliation in T_1 admitting a generic meromorphic first integral is equivalent to one given by polynomial equations.

The genericity condition can be stated as follows: a meromorphic first integral exists that vanishes with multiplicity one on the unique separatrix that is tangent to the exceptional divisor. This theorem improves the result in [2] in the generic case. In [2], Casale proves that any germ admitting a meromorphic first integral is equivalent to a germ of a foliation on an algebraic surface (not necessarily the projective plane). By using the results in [1] we are able to extend his result to foliations in T defined by meromorphic functions. In order to state the theorem along the same lines of

[2], we use the notion of **algebraic-like foliation**: we say that a germ of holomorphic foliation \mathcal{F} is an **algebraic-like foliation** when there exists an algebraic surface S and an algebraic foliation of S which is equivalent to \mathcal{F} in a neighborhood of some singularity.

Theorem 3. Any foliation in T admitting a meromorphic first integral is an algebraic-like foliation.

A good point to discuss is whether Theorem 3 can be stated replacing "algebraic-like" foliation by "polynomially defined" foliation. We do not know the answer in general.

Sections 2, 3 e 4 of the present paper are devoted to the proof of Theorem 1, and Sections 5 and 6 to prove Theorems 2 and 3. The two blocks are independent and can be read separately.

We thank G. Smith for useful conversations and especially for the example of the quintic in section 6.1.

2 Preliminaries

We consider the set T_1 of germs of holomorphic foliations in $(\mathbb{C}^2, 0)$ defined by differential 1-forms of the type

$$(1) \quad \sum_{j \geq 2} b_j(x, y) dx - \sum_{j \geq 2} a_j(x, y) dy = 0$$

where $a_2(x, y) = xy$, $b_2(x, y) = y^2$ and $xb_3(x, y) - ya_3(x, y) = \beta x^4$, $\beta \neq 0$.

After one blow-up $(x, t) \mapsto (x, tx)$, the foliation is regular, with only one point of tangency of order 1 with the exceptional divisor (the equation is normalized as to have the tangency point given by $t = 0$).

To each $\mathcal{F} \in T_1$ we can associate a local involution $i_{\mathcal{F}}(t)$ defined for $t \in \mathbb{C}$ close to $0 \in \mathbb{C}$; moreover, it can be easily seen that for a holomorphic family $\alpha \in U \subset \mathbb{C}^m \mapsto \mathcal{F}_{\alpha} \in T_1$, the function $(\alpha, t) \mapsto i_{\mathcal{F}_{\alpha}}(t)$ is holomorphic.

Let $Inv := \{i(t) = \sum_{j \geq 1} a_j t^j \in \mathbb{C}\{t\}, a_1 = -1, i \circ i(t) = t\}$; we consider in $\mathbb{C}\{t\}$ the norm $\|\sum_{j \geq 0} c_j t^j\| := \sum_{j \geq 0} \frac{|c_j|}{j!}$, which induces a distance d . Since

$$(2) \quad Inv_k := \{i(t) \in \mathbb{C}\{t\}; i(0) = 0, i'(0) = -1 \text{ and } i \circ i(t) = t \bmod t^{k+1}\}$$

is closed in $(\mathbb{C}\{t\}, d)$ for each $k \geq 1$, and $Inv = \cap_{k \geq 1} Inv_k$, we conclude that Inv is closed in $(\mathbb{C}\{t\}, d)$.

Now we take

$$(3) \quad \mathcal{L}_1\{t\} := \left\{ \sum_{j \geq 0} c_j t^j \in \mathbb{C}\{t\}; \sum_{j \geq 0} |c_j| < \infty \right\}$$

Clearly $\mathcal{L}_1\{t\}$ is a vector subspace of $\mathbb{C}\{t\}$; any power series in $\mathcal{L}_1\{t\}$ defines a holomorphic function whose domain of definition contains the unit disc $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$. On the other hand, the Taylor series centered at $0 \in \mathbb{C}$ of a holomorphic function defined in a neighborhood of $\bar{\mathbb{D}}$ belongs to $\mathcal{L}_1\{t\}$.

We define $\|\sum_{j \geq 0} c_j t^j\|_1 := \sum_{j \geq 0} |c_j|$ for $\sum_{j \geq 0} c_j t^j \in \mathcal{L}_1\{t\}$; with this norm $\mathcal{L}_1\{t\}$ becomes a Banach space. Let d_1 be the associated distance.

Lemma 1. The inclusion map from $(\mathcal{L}_1\{t\}, d_1)$ to $(\mathbb{C}\{t\}, d)$ is continuous.

Proof. It is enough to remark that

$$(4) \quad \left\| \sum_{j \geq 0} c_j t^j \right\| = \sum_{j \geq 0} \frac{|c_j|}{j!} \leq \sum_{j \geq 0} |c_j| = \left\| \sum_{j \geq 0} c_j t^j \right\|_1$$

□

It follows that $Inv \cap \mathcal{L}_1\{t\}$ is closed in $(\mathcal{L}_1\{t\}, d)$. Therefore, $Inv \cap \mathcal{L}_1\{t\}$, endowed with the metric d_1 , becomes a complete metric space, in particular a Baire space.

3 Realizing Involutions

We introduced in the last section a map i that takes foliations of T_1 to involutions of $\mathbb{C}\{t\}$.

Lemma 2. The map $i : T_1 \longrightarrow Inv$ is surjective.

Proof. 1) Given some $i(t) \in Inv$, we construct first a local foliation around the disc $\mathbb{D} \times \{0\}$ which has a tangency point at $(0, 0)$ with this disc and whose associated involution is $i(t)$. We start by mapping $\mathbb{D} \times 0$ to $\mathbb{C} \times 0$ via some holomorphic diffeomorphism ϕ which satisfies

- $\phi(0) = 0$.
- ϕ conjugates $t \mapsto -t$ to $i(t)$.

We then extend ϕ to some holomorphic diffeomorphism Φ in a neighborhood of $\mathbb{D} \times \{0\}$ and define the foliation \mathcal{H} as the image by Φ of the foliation defined as $d(x - t^2) = 0$.

2) The next step consists in the following gluing process:

- we take the surface S obtained after blowing-up \mathbb{C}^2 at $(0, 0)$, foliated by $dt = 0$ ((x, y) are coordinates in \mathbb{C} , $(t = \frac{y}{x}, x)$ are coordinates in S). In S we remove a disc $\mathbb{D}_{\frac{1}{2}} \times U$, where U is a small neighborhood of $0 \in \mathbb{C}$.
- the trivial foliation $dt = 0$ in $\overline{(\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}})} \times U$ is equivalent to the restriction of \mathcal{H} to a region R given as a local saturation (along the leaves of \mathcal{H}) of some annulus $\mathbb{A} \times \{0\}$ around $(0, 0) \in \mathbb{D} \times \{0\}$. This equivalence is then used to glue $\mathcal{G}|_{S \setminus \{(\overline{\mathbb{D} \setminus \mathbb{D}_{\frac{1}{2}})} \times U\}}$ with \mathcal{H} ; since it can be taken close to the Identity, the resulting foliation is defined around a (-1) -curve and is thus equivalent to the blow-up of an element of T_1 . \square

4 Adapting Genzmer-Teyssier

Our aim is to show that there are foliations in T_1 which are not holomorphically equivalent to any foliation in T_1 defined by a polynomial equation. In order to do that, we need to change the map i . Let G be the group of Moe-bius transformations of \mathbb{P}^1 which fix $0 \in \mathbb{C}$ (in the t -coordinate associated to the blow up). We consider the map

$$(5) \quad I : G \times T_1 \longrightarrow Inv, \quad I(g, \mathcal{F}) = g^{-1} \circ i_{\mathcal{F}} \circ g.$$

Remark: In fact the map of Lemma 2 induces a bijection between $[T_1]$ and Inv/G (see [1]).

Let $T_1^{(k)}$ denote the subset of elements of T_1 defined by a polynomial equation of degree k . The goal is therefore to prove that

$$(6) \quad \cup_k I(G \times T_1^{(k)}) \neq Inv$$

We follow the procedure exposed in [5]. We have to prove that the image of an embedding $\xi : \overline{\mathbb{D}}^l \rightarrow (Inv, d)$ leaves a trace in $Inv \cap \mathcal{L}_1\{t\}$ which has empty interior in the topology defined by d_1 .

Let us consider then some $f \in Im(\xi) \cap \mathcal{L}_1\{t\}$ and $0 < \lambda < 1$. Any power series defined as $f_\lambda(t) = \lambda^{-1}f(\lambda t)$ belongs to $\mathcal{L}_1\{t\}$ and $d_1(f_\lambda, f) \rightarrow 0$ as $\lambda \rightarrow 1$; furthermore, the radius of convergence of f_λ is greater than 1. If for some sequence $\lambda_m \rightarrow 1$ it happens that $f_{\lambda_m} \notin Im(\xi)$, we are done; otherwise we replace f by some d_1 -close $f_{\bar{\lambda}}$ and we still have $f_{\bar{\lambda}} \in Im(\xi) \cap \mathcal{L}_1\{t\}$. In order to simplify the notation we use f instead of $f_{\bar{\lambda}}$.

We then have $f = -t + \sum c_j t^j \in Im(\xi) \cap \mathcal{L}_1\{t\}$, with radius of convergence greater than 1. The tangent space $T_f Im(\xi)$ has some finite dimension l . Any element in $T_f Im(\xi)$ is a power series $\sum a_j t^j \in \mathbb{C}\{t\}$; after truncating the elements of $T_f Im(\xi)$ up to some sufficiently high order m_0 , we still have a linear subspace of dimension l . Therefore, for each $m \geq m_0$, a power series in $T_f Im(\xi)$ is completely determined once we know the first m coefficients.

Now we consider the path $\alpha(u) := h_u^{-1} \circ f \circ h_u$, where $h_u(t) = t + ut^m$ for $m \geq 0$. Clearly h_u^{-1} is well defined in some disc of radius greater than 1 for $|u|$ small enough. This guarantees that $\alpha(u)$ is inside $Inv \cap \mathcal{L}_1\{t\}$. The tangent vector $\alpha'(0)$ (*which we intend to prove that is transverse to $T_f Im(\xi)$*) has its $(m-1)$ -jet equal to zero, therefore $\alpha'(0) = 0$ if it belongs to $T_f Im(\xi)$. But an easy computation shows that

$$(7) \quad \alpha(u)(t) = h_u^{-1} \circ f \circ h_u(t) = -t + \sum_{j=2}^{m-1} c_j t^j + (c_m - 2u)t^m + \dots$$

and then

$$(8) \quad \alpha'(0) = -2t^m + \dots$$

which is a contradiction that proves Theorem 1.

5 A Model

Let us consider a holomorphic foliation \mathcal{G} defined in some open set of \mathbb{C}^2 which contains $0 \in \mathbb{C}^2$ as an isolated singularity; we assume that the exceptional divisor is not invariant for the blown-up foliation. We will now conjugate \mathcal{G} to a special model \mathcal{F} .

- **Step 1:** we blow up at $0 \in \mathbb{C}^2$; the exceptional divisor E_1 is not invariant for the blown-up foliation $\tilde{\mathcal{G}}_1$. We select a point $p \in E_1$ where $\tilde{\mathcal{G}}_1$ is transverse to E_1 and take a neighborhood V_1 of this point where $\tilde{\mathcal{G}}_1$ is trivial. In parallel, we blow up at $0 \in \mathbb{C}^2$ the trivial foliation $dy = 0$ to a foliation $\tilde{\mathcal{G}}_2$ which now has the exceptional divisor E_2 as an invariant set (with one singularity). We take a regular point of $\tilde{\mathcal{G}}_2$ in E_2 and a neighborhood V_2 of this point where $\tilde{\mathcal{G}}_2$ is trivial. We then glue $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$ by a holomorphic diffeomorphism from V_1 to V_2 which sends $\tilde{\mathcal{G}}_1|_{V_1}$ to $\tilde{\mathcal{G}}_2|_{V_2}$. We get a surface which contains two divisors, still denoted by E_1 and E_2 , with $E_1 \cdot E_1 = E_2 \cdot E_2 = -1$ and $E_1 \cdot E_2 = 1$, and a foliation $\tilde{\mathcal{G}}$ conjugated to $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ in neighborhoods of E_1 and E_2 respectively.
- **Step 2:** we consider now the surface obtained after blowing up $\mathbb{D} \times \mathbb{P}^1$ at some point of $\{0\} \times \mathbb{P}^1$; we have inside it two divisors E'_1 and E'_2 such that $E'_1 \cdot E'_1 = E'_2 \cdot E'_2 = -1$ and $E'_1 \cdot E'_2 = 1$. Since a neighborhood of $E_1 \cup E_2$ is biholomorphically equivalent to a neighborhood of $E'_1 \cup E'_2$ by a diffeomorphism that takes E_1 to E'_1 and E_2 to E'_2 , we may define a foliation $\tilde{\mathcal{F}}$ in a neighborhood of $E'_1 \cup E'_2$ as the image of $\tilde{\mathcal{G}}$. The blow-down of the restriction of $\tilde{\mathcal{F}}$ to a neighborhood of E'_1 is the model \mathcal{F} we mentioned above. We may see $\tilde{\mathcal{F}}$ as the blow-up at some point of $\{0\} \times \mathbb{P}^1$ of a foliation \mathcal{F}_1 defined in $\mathbb{D} \times \mathbb{P}^1$.

In other words, modulo holomorphic equivalence, \mathcal{G} is obtained by blowing-up a foliation defined in $\mathbb{D} \times \mathbb{P}^1$ at some point of transversality with $\{0\} \times \mathbb{P}^1$. As can be checked there are many choices involved in the construction and the model obtained in the product is not unique at all. Our strategy to find extensions to algebraic foliations will be to extend these types of foliated products to some foliation on $S \times \mathbb{P}^1$ where S is a compact Riemann surface.

6 Algebraic Case

Let us remark that if $\mathcal{G} \in T$ then \mathcal{F}_1 is regular along $\{0\} \times \mathbb{P}^1$ as well (we are keeping the same notation used in the last Section). Furthermore, if \mathcal{G} has a meromorphic first integral then the same is true for \mathcal{F}_1 ; in particular a first integral $R(x, t)$ can be seen as a holomorphic family of rational functions $x \in \mathbb{D} \mapsto R_x(t) = R(x, t) \in \mathbb{P}^1$ of some degree d . It is not difficult to show that $x \in \mathbb{D} \mapsto R_x$ is locally injective at $x = 0$.

To prove Theorem 2 we are going to see that up to reparametrizing the x -variable, we can extend this family to one parametrized by \mathbb{P}^1 . The induced foliation will then define an extension of \mathcal{F}_1 to $\mathbb{P}^1 \times \mathbb{P}^1$. To prove Theorem 3 what we are going to do is approximate this family by one passing through $R_0(t)$ and whose parameter space is a compact Riemann surface S . The associated foliation on $S \times \mathbb{P}^1$ —an algebraic surface—can be blown up at an adequate point of $\{0\} \times \mathbb{P}^1$ so that the obtained foliation approximates $\tilde{\mathcal{F}}$ and will satisfy the necessary conditions to be conjugated to $\tilde{\mathcal{F}}$ in a neighborhood of E'_1 (according to [1]).

6.1 Proof of Theorem 2

The proof of Theorem 2 does not use approximation and can be done after a suitable change of the first integral and of the coordinates on the product $\mathbb{D} \times \mathbb{P}^1$. In this subsection we suppose that \mathcal{G} is a foliation in T_1 admitting a meromorphic first integral and consider its model \mathcal{F}_1 and a first integral R meromorphic on $\mathbb{D} \times \mathbb{P}^1$, satisfying that $R^{-1}(0)$ contains a component that is tangent to the central fiber $E = \{0\} \times \mathbb{P}^1$ at the point $(0, 0)$. By the genericity condition on R we have that $R_0(t) = R(0, t)$ has a simple critical point at 0. Remark that if we post-compose $R(x, t)$ with a non-constant rational function Q on \mathbb{P}^1 , the level sets of $Q \circ R$ still define the same foliation. By choosing Q and the x coordinate appropriately we claim that we can suppose that the first integral R for \mathcal{F}_1 satisfies

1. For any critical *value* $v \neq 0$ of R_0 except possibly for one of them, there is a connected component of $R^{-1}(v)$ that is not critical for R , intersecting $0 \times \mathbb{P}^1$ on two points $q, h(q)$ where h is the involution associated to \mathcal{F}_1 at $(0, 0)$.
2. $(x, 0) \in \mathbb{D} \times \mathbb{P}^1$ is a simple critical point of $R_x(t) = R(x, t)$ with critical value $R_x(0) = x$.

To prove that condition 1. can be attained, take a domain D where $h : D \rightarrow D$ is conjugated to a rotation and each leaf cutting $D \setminus 0$ is a disc intersecting D on two points. Take a round disc $D_r \subset R_0(D)$ containing 0. By composing R_0 with a Moebius transformation we can suppose that $D_r = \mathbb{H}$, the upper half plane in \mathbb{C} , and the critical values $v_1, \dots, v_k \in \mathbb{C} \setminus R_0(D)$ of R_0 belong to a small neighbourhood of ∞ .

Next take a polynomial $Q(z) = z^5 + a_4 z^4 + \dots + a_1 z + a_0$ with real coefficients $a_i \in \mathbb{R}$ satisfying that its four critical points $c_1 < c_2 < c_3 < c_4$ in \mathbb{C} lie in \mathbb{R} , and the equation $Q(z) = Q(c_i)$ has precisely two distinct real roots for each $i = 1, \dots, 4$. By construction the other two roots of each such equation are complex conjugate. In particular all finite critical values of Q are attained at regular points in \mathbb{H} . To show that the finite critical values of $Q \circ R_0$ are also attained in D it suffices to remark that in a neighbourhood $U_\rho = \{z \in \mathbb{H} : |z| > \rho\}$ for ρ sufficiently big p acts like $z \mapsto z^5$ and thus $Q(U_\rho)$ covers a pointed neighbourhood of infinity. As $Q(v_i)$ are close to ∞ we have that $Q(v_i) \subset Q(\mathbb{H})$.

Once condition 1 is satisfied, condition 2 can be obtained by a change of variables. Indeed, if R already satisfies 1 then in some connected and simply connected neighbourhood $U \subset 0 \times \mathbb{P}^1$ where the involution associated to \mathcal{F}_1 is defined, we can define two degree two branched coverings whose fibers coincide: on the one hand $R|_U$ and on the other, the projection π along the leaves of the foliation from U to the set $t = 0$. Hence we can parametrize a neighbourhood of 0 in $t = 0$ by the values $x \in W = R(U)$ and then on $W \times \mathbb{P}^1$ the map $(x, t) \mapsto \tilde{R}(x, t) := R_{\pi \circ R^{-1}(x)}(t)$ is a holomorphic family of rational functions and satisfies $\tilde{R}(x, 0) = x$. By construction W contains all the critical values of \tilde{R} except maybe for one. The foliation defined by the levels of \tilde{R} is equivalent to \mathcal{F}_1 . After applying the Riemann mapping theorem to W we can suppose $W = \mathbb{D}$.

Let $C = \{v_1, \dots, v_k\} \subset \mathbb{P}^1 \setminus 0$ be the set of critical values of R_0 different from 0. By construction, for each $x \in \mathbb{D} \setminus (C \cap \mathbb{D})$ the rational function R_x has degree d and has critical values at $\{x\} \cup C$. Indeed, since the tangency point is simple and unique, there is a unique component of the tangency divisor between \mathcal{F}_1 and the vertical fibration, and it corresponds to the set $t = 0$ by construction. Each other critical value of R_0 produces a critical value of R_x at the point of intersection of the corresponding leaf with the fibre $\{x\} \times \mathbb{P}^1$. The restriction of R_x to $R_x^{-1}(\mathbb{P}^1 \setminus C \cup \{x\})$ defines a topological degree d covering having monodromy in a conjugacy class of a subgroup G_x of the

symmetric subgroup in d symbols. By continuity the class of G_x is constant G for all $x \in \mathbb{D} \setminus C$. By connectedness of the covering we know that G acts transitively on each fibre.

Let \mathcal{H} be the Hurwitz space associated to the triple $(d, k+1, G)$, that is, the space of isomorphism classes of topological coverings of the sphere minus $k+1$ points having degree d and monodormy conjugated to G . Two coverings X, X' are isomorphic if there exists a homeomorphism between the covering spaces $H : X \rightarrow X'$ such that $\pi = \pi' \circ H$, where π, π' denote the covering projections. In particular for two coverings to be equivalent they need to omit the same set of values on the sphere. Let \mathcal{V} be the set of unordered $(k+1)$ -uples of distinct points in \mathbb{P}^1 . Hurwitz (see [6] or [4]) showed that the projection $P : \mathcal{H} \rightarrow \mathcal{V}$, defined by associating to any class of coverings the set of values it omits on the sphere, is itself a topological covering map. We have a natural, continuous, non-constant map $f : \mathbb{D} \setminus C \rightarrow \mathcal{V}$ defined by $f(x) = P([R_x])$. If we take the coordinates in \mathbb{P}^1 we took before it can be written as $f(x) = [\{x, v_1, \dots, v_k\}] \in \mathcal{V}$ and it extends naturally to a map $f : \mathbb{P}^1 \setminus C \rightarrow \mathcal{V}$ that is actually holomorphic. To lift f to a map $F : \mathbb{P}^1 \setminus C \rightarrow \mathcal{H}$ continuously it suffices to guarantee that at the fundamental group level we have the inclusion $\text{Im} f_* \subset \text{Im} P_*$. This condition is satisfied since we can find generators $\gamma_1, \dots, \gamma_{k-1}$ of the fundamental group of $\mathbb{P}^1 \setminus C$ whose images lie in $\mathbb{D} \setminus (C \cap \mathbb{D})$, and thus the loops $t \mapsto f(\gamma_i(t))$ in \mathcal{V} lift to loops $t \mapsto [R_{\gamma_i(t)}]$ in \mathcal{H} . The resulting F has finite fibers and is holomorphic when we consider the unique complex structure on \mathcal{H} for which P is holomorphic (recall that \mathcal{V} already carries a holomorphic structure).

For each $x \in \mathbb{P}^1 \setminus C$, $F(x)$ defines a unique branched covering of the whole sphere, and hence can be considered as a rational function of degree d . The new holomorphic map $\mathbb{P}^1 \setminus C \rightarrow \text{Rat}_{\leq d}$ so defined is holomorphic and has finite fibres. Hence it has no essential singularity and it extends to a holomorphic map $\mathbf{F} : \mathbb{P}^1 \rightarrow \text{Rat}_{\leq d}$.

By construction and uniqueness of complex structure on the sphere, there exists for each $x \in \mathbb{D} \setminus C$ a Moebius transformation H_x such that $R_x \circ H_x = F(x)$. In particular, by pulling R back by the change of coordinates $(x, t) \mapsto (x, H_x(t))$ defined in a neighbourhood of $x = 0$ we have that the germ of $x \mapsto \mathbf{F}(x)$ at 0 describes the pull back of the foliation \mathcal{F}_1 . This foliation extends to $\mathbb{P}^1 \times \mathbb{P}^1$ by the level sets of $F(x, t) = \mathbf{F}(x)(t)$. By blowing up a point of transversality of the foliation and the central fibre and contracting the strict transform of the fibre we obtain a foliation in $\mathbb{P}^1 \times \mathbb{P}^1$ having a singularity in T_1 with the same holonomy involution as \mathcal{G} modulo

conjugation by the Moebius transformation H_0 . By the Remark of Section 4 two foliations in T_1 having the same involution modulo conjugation by a Moebius transformation are analytically equivalent. Hence we have that the germ \mathcal{G} is equivalent to the germ of that singularity. The obtained foliation is obviously defined by polynomial equations.

This proof cannot be extended to other foliations in T in general because there appear many components of the tangency divisor between \mathcal{F}_1 and the vertical foliation and there is no way of finding a coordinate where all the curves of critical values can be extended in the same parametrization to \mathbb{P}^1 . Even if the extension existed there would be intersections of the parametrized curves of critical values and we would have no control over the monodromies around those intersection points. It is for this reason that we will change our point of view and, instead of trying to extend the germ of curve $x \mapsto R_x$ we will try to approximate it by one that extends and use that for good approximations the associated foliations are locally equivalent.

6.2 Critical Points

We start by analysing the curves of critical points of $x \mapsto R_x(t)$. We will assume that there exists a fixed neighborhood U (independent of x) of $\infty \in \mathbb{P}^1$ such that no critical point is inside this neighborhood. We have the following possibilities:

- (i) the leaf of \mathcal{F}_1 that passes through a critical point of $R_0(t)$ (of order $m \in \mathbb{N}$) is transversal to $\{0\} \times \mathbb{P}^1$; we parametrise the leaf as $x \mapsto (x, f(x))$. Since the first integral assumes a constant value along each nearby leaf, we see that each point $(x, f(x))$ is also a critical point of order m of $R_x(t)$. Consequently the curve $t - f(x)$ is contained in the singular set of the foliation defined by $dR = 0$; we call such a curve of critical points (or singular points) a *level type curve*. We may write locally (assuming $t_0 = 0$ for simplicity) that

$$R(x, t) = a + (t - f(x))^{m+1}h(x, t)$$

where $a \in \mathbb{C}$, $h(0, 0) \neq 0$. Therefore

$$dR = [(m+1)(t - f(x))^m h + (t - f(x))^{m+1} \frac{\partial h}{\partial x}] dx$$

$$+[-(m+1)(t-f(x))^m h g' + (t-f(x))^{m+1} \frac{\partial h}{\partial t}] dt$$

The foliation $dR = 0$ has $(t-f(x))^m$ as its equation of zeroes. The equation of \mathcal{F}_1 is then $\frac{dR}{(t-f(x))^m} = 0$.

- (ii) the critical point $(0, t_0)$ is a point of tangency of \mathcal{F}_1 with $\{0\} \times \mathbb{P}^1$; it gives rise to a curve of critical points of $R_x(t)$, or points of tangency between \mathcal{F}_1 and the vertical lines $x = \text{const}$, which crosses $\{0\} \times \mathbb{P}^1$ at the point $(0, t_0)$ (we put again $t_0 = 0$). The foliation \mathcal{F}_1 is obtained in a neighborhood of $(0, 0)$ once we divide $dR = 0$ by the equation of its zeroes. If a component of the curve of critical points is invariant by \mathcal{F}_1 , it necessarily coincides with the leaf which is tangent to $\{0\} \times \mathbb{P}^1$ at $(0, 0)$; we call it also a *level type curve* of critical points (of some order M). It has as equation $x - g(t) = 0$, where $g(t) = t^{l+1} \tilde{g}(t)$ with $l \geq 1$ and $\tilde{g}(0) \neq 0$. We apply the same argument as in case (i) to a neighborhood of a point of this curve for which $x \neq 0$ and conclude that $(x - g(t))^M = 0$ is inside the set of zeroes of dR (a fortiori in a neighborhood of $(0, 0)$ as well).

Now let us analyse the case of a component of a *non-invariant curve* of critical points, that is, one that is not \mathcal{F}_1 -invariant. We observe that the zeroes of dR are inside the zeroes of $\frac{\partial R}{\partial t} = 0$. Locally at a point where $x \neq 0$ we have

$$R(x, t) = a(x) + (t - u(x))^{l+1} h(x, t)$$

where $a(x)$ is not constant (otherwise we would have case (i)), $h(0, 0) \neq 0$, $l \geq 1$ and $t - u(x) = 0$ is the local equation of the component. It follows from

$$\begin{aligned} dR &= [a'(x) - (l+1)(t-u(x))^l f'(x)h + (t-u(x))^{l+1} \frac{\partial h}{\partial x}] dx \\ &+ [(l+1)(t-u(x))^l h + (t-u(x))^{l+1} \frac{\partial h}{\partial t}] dt \end{aligned}$$

that the coefficients of dx and dt have no common factors; therefore there is no new curve of zeroes arising from the type of curve of critical points under consideration. We conclude that in a neighborhood of $(0, 0)$ we just have to take $\frac{dR}{(x-g(t))^M} = 0$ in order to define \mathcal{F}_1 . Of course it may happen that the leaf of \mathcal{F}_1 which is tangent to $\{0\} \times \mathbb{P}^1$ is not a level type curve of critical points.

We may summarise this information about the zeroes of dR as follows:

- there are curves of level type $x \mapsto f_1(x), \dots, f_k(x)$ which correspond to critical points of orders m_1, \dots, m_k ; these curves are transversal to $\{0\} \times \mathbb{P}^1$, and locally $R(x, t) = a_j + (t - f_j(x))^{m_j+1} h_j(x, t)$. Locally at each of these critical points the foliation \mathcal{F}_1 is given by the equation $\frac{dR}{(t-f_j(x))^{m_j}} = 0$.
- there are $(l_1 + 1), \dots, (l_s + 1)$ -valued curves $x \mapsto P_1(x), \dots, P_s(x)$ of critical points of orders M_1, \dots, M_s which are curves of level type (for $x \neq 0$); each curve $P'_j = \cup_x P_j(x)$ is tangent to $\{0\} \times \mathbb{P}^1$ in order l_j at a critical point of R_0 ; its equation is $x - g_j(t) = 0$ with $g_j(t) = t^{l_j+1} \tilde{g}_j(t)$, $l \geq 1$ and $\tilde{g}_j(0) \neq 0$. Locally at each of these critical points the foliation \mathcal{F}_1 is given by the equation $\frac{dR}{(x-g_j(t))^{M_j}} = 0$; we have also $R \equiv A_j$ along each curve $x - g_j(t) = 0$.

6.3 Proof of Theorem 3

Now we will consider the algebraic variety which is the closure of the space of degree d rational functions of \mathbb{P}^1 which have the configuration of critical points we presented, namely:

- * the rational function has values a_1, \dots, a_k at critical points which have orders m_1, \dots, m_k respectively.
- ** the rational function has values A_1, \dots, A_s at $(l_1+1), \dots, (l_s+1)$ critical points which have orders M_1, \dots, M_s respectively.

Let us denote also by R the curve given by $R(x) = R_x$; it belongs to a smooth stratum B of this variety for $x \neq 0$ small and $R(0)$ belongs to \bar{B} , which is also an algebraic variety. Let π be a desingularisation of \bar{B} and of R at the point $R(0)$. The strict transform \tilde{R} of R crosses the boundary of $\pi^{-1}(B)$ at a smooth point $\tilde{R}(0) \in \pi^{-1}(\bar{B})$. We have a foliation in $\tilde{R} \times \mathbb{P}^1$ given by the level curves of the meromorphic function $(\tilde{p}, t) \mapsto R_{\pi(\tilde{p})}(t)$, which is conjugated to \mathcal{F}_1 (since $x \in \mathbb{D} \mapsto R_x$ is injective, it works as a desingularization of the curve R).

Next we take an algebraic curve \tilde{S} in $\pi^{-1}(\bar{B})$ which passes through $\tilde{R}(0)$ smoothly with order of tangency N as big as we wish with \tilde{R} at the point

$\tilde{R}(0)$; the choice of N will depend on the statements which will follow. Consequently in \bar{B} we may choose an algebraic family S of rational functions parametrized by a map of $x \in \mathbb{D}$ near the point $S(0) = R(0)$ such that both associated foliations $dR = 0$ and $dS = 0$ are as close as we wish in $\mathbb{D} \times \mathbb{P}^1$ (in fact, we need to cover $\mathbb{D} \times \mathbb{P}^1$ by two coordinates systems; in the chart that contains $\{0\} \times \{\infty\}$ we use $R = \text{const}$ and $S = \text{const}$ to define the associated foliations, which are both regular ones; in the chart that contains $\{0\} \times \{0\}$ the foliations $dR = 0$ and $dS = 0$ are singular).

Next we need to prove that after eliminating the singularities of $dS = 0$ we obtain a foliation which is regular and has the same type of tangencies with $\{0\} \times \mathbb{P}^1$ as \mathcal{F}_1 .

Let us fix a family of disjoint polydiscs, one for each critical point of $R_0 = S_0$. If $(0, t_j)$ is a critical point, we take $\Delta_j = \{(x, t); |x| \leq \epsilon, |t - t_j| \leq \epsilon\}$. If S_x is sufficiently close to R_x and ϵ is small, the configuration of critical points of S_x in each set $K_j = \{(x, t); \frac{\epsilon}{2} \leq |x| \leq \epsilon, |t - t_j| \leq \epsilon\}$ is the same as the configuration of R_x . This means that for S_x we have in each K_j :

- there are new curves of level type $x \mapsto \hat{f}_1(x), \dots, \hat{f}_k(x)$ which correspond to critical point of orders m_1, \dots, m_k ; S takes the values a_1, \dots, a_k along these curves.
- there are new $(l_1 + 1), \dots, (l_s + 1)$ -valued curves $x \mapsto \hat{P}_1, \dots, \hat{P}_s(x)$ which are curves of level type corresponding to critical points of orders M_1, \dots, M_s ; S takes the values A_1, \dots, A_s along these curves.

Since the set of critical points of S_x inside each Δ_j is an analytic curve, we conclude that the critical curve of level type that lies in D_j has an extension which passes through the point $(0, t_j)$ and reproduces the same type of the corresponding critical curve of R_x . The singular set of the foliations $dR = 0$ and $dS = 0$ (which appear along the curves of critical points of level type because of condition **) are then sufficiently close (if R_x and S_x are sufficiently close) to allow us to conclude that the regular foliations associated to them (after elimination of singularities) are also close. Notice that the equality $R_0 = S_0$ implies that the involutions associated to both foliations coincide.

Now we blow-up the point $(0, \infty) \in \mathbb{D} \times \mathbb{P}^1$; we obtain two foliations (one for R_x and the other one for S_x) defined in neighborhoods of the strict transform E'_1 of $\{0\} \times \mathbb{P}^1$ that have the same holonomy invariants at the

points of tangency with E'_1 since $R_0 = S_0$. Furthermore, they may be assumed to be close enough as to allow application of [1]; consequently they are conjugated.

We remark that we have a global foliation associated to the strict transform \tilde{S} of S (the closure of $\pi^{-1}(S \setminus \{S(0)\})$). We define in $\tilde{S} \times \mathbb{P}^1$ the meromorphic function which at a fiber $\{p\} \times \mathbb{P}^1$ is exactly \tilde{S}_p ; we have then inside the algebraic variety $\tilde{S} \times \mathbb{P}^1$ the algebraic foliation given by the level curves of this meromorphic function (it is easy to see that this foliation is equivalent to $dS = 0$ in a neighborhood of $\pi^{-1}(S(0)) \times \mathbb{P}^1$). Now we just blow up $\tilde{S} \times \mathbb{P}^1$ at the point $(0, \infty) \in \mathbb{D} \times \mathbb{P}^1$, and conclude by blowing down the exceptional divisor E'_1 .

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